Journal of Statistical Physics, Vol. 87, Nos. 1/2, 1997

Relaxed Criteria of the Dobrushin–Shlosman Mixing Condition

Nobuo Yoshida[1]

Received April 17, 1996; final October 9, 1996

An interacting particle system (Glauber dynamics) which evolves on a finite subset in the d-dimensional integer lattice is considered. It is known that a mixing property of the Gibbs state in the sense of Dobrushin and Shlosman is equivalent to several very strong estimates in terms of the Glauber dynamics. We show that similar, but seemingly much milder estimates are again equivalent to the Dobrushin–Shlosman mixing condition, hence to the original ones found by Stroock and Zegarlinski. This may be understood as the absence of intermediate speed of convergence to equilibrium.

KEY WORDS: Relaxed criteria; the Dobrushin–Shlosman mixing condition; Glauber dynamics; convergence to equilibrium.

1. INTRODUCTION

A standard way to describe a mixing property of a Gibbs state is to estimate the difference of the expectations of a local observable with respect to a finite-volume Gibbs state with different boundary conditions. For example, the Dobrushin–Shlosman mixing condition which we will discuss can roughly be stated as follows: the difference mentioned above is exponentially small in the distance between the support of the observable and the sites at which the boundary conditions are different [cf. (2.16)]. On the other hand, since the mixing property of a Gibbs state reflects the rapid relaxation of the Glauber dynamics, the mixing property can be expressed in the following different ways in terms of the Glauber dynamics: (a) estimate of the logarithmic Sobolev constant, (b) estimate of the spectral gap, (c) estimate of the rate of convergence (the difference between the

[1] Division of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-01, Japan; e-mail: nobuo@kusm.kyoto-u.ac.jp.

0022-4715/97/0400-0293$12.50/0

semigroup at time t and its equilibrium). Stroock and Zegarlinski[16] succeeded in rephrasing the Dobrushin–Shlosman mixing condition in three ways described above, which we will review as Theorem 3.1 below.

The purpose of this paper is to present relaxed criteria of the Dobrushin–Shlosman mixing condition of types (a)–(c), which are respectively conditions (2a)–(2c) in Theorem 3.2, our main result. Technically, the derivation of the original Dobrushin–Shlosman mixing condition from these relaxed criteria is based on the fact that the exponential decay in the statement of the Dobrushin–Shlosman mixing condition is equivalent to a certain polynomial decay. This point is also made clear as conditions (2d) and (2e) in Theorem 3.2. These relaxed criteria are potentially useful to check the Dobrushin–Shlosman mixing condition, which is often accepted as an assumption to do something with, but is not always easy to verify in practical applications. Since our result is based on the very old and well-known idea that "an exponential mixing follows from a certain polynomial mixing," there are many results with the same sprit in the earlier literature. The relation between these earlier results and ours will be discussed in the series of remarks following the statement of Theorem 3.2.

2. BASIC DEFINITIONS

The lattice. We will work on the d-dimensional integer lattice $\mathbf{Z}^d = \{x = (x_i)_{i=1}^d : x_i \in \mathbf{Z}\}$, on which we consider the l^∞-norm; $|x| = \max_{1 \leqslant i \leqslant d} |x_i|$. For a set $\Lambda \subset \mathbf{Z}^d$, diam Λ and $|\Lambda|$ stand, respectively, for its diameter and the number of the points it contains. We write $\Lambda \subset\subset \mathbf{Z}^d$ when $1 \leqslant |\Lambda| < \infty$ and define a family $\mathscr{A}$ by

$$\mathscr{A} = \{\Lambda; \Lambda \subset\subset \mathbf{Z}^d\} \tag{2.1}$$

The distance between two subsets Λ_1 and Λ_2 of $\mathbf{Z}^d$ will be denoted by $d(\Lambda_1, \Lambda_2)$. For $r \geqslant 1$, the r-boundary of a set Λ is defined by

$$\partial_r \Lambda = \{x \notin \Lambda; d(x, \Lambda) \leqslant r\} \tag{2.2}$$

The value of r will eventually be chosen as an upper bound r_0 of the range of the interaction we consider [cf. (2.5) below]. For $v \in \mathbf{Z}^d$ and an integer $m \geqslant 1$, we define a subset $\mathscr{B}_v(m)$ of $\mathscr{A}$ as the totality of $\Lambda \subset\subset \mathbf{Z}^d$ of the following form:

$$\Lambda = \{x \in \mathbf{Z}^d; v_i + ma_i \leqslant x_i < v_i + mb_i \text{ for } i = 1,..., d\} \tag{2.3}$$

where a_i and b_i are integers with $a_i < b_i$. In particular, the class $\mathscr{B}_v(m)$ consists of boxes with every sidelength a positive multiple of m. We define a subset $\mathscr{C}_v(m)$ of $\mathscr{B}_v(m)$ as the totality of cubes in $\mathscr{B}_v(m)$, i.e., the totality of $\Lambda \subset\subset \mathbf{Z}^d$ of the form (2.3) with $b_i - a_i$ $(1 \leqslant i \leqslant d)$ identical.

The configurations. We take a finite set S as a single spin space and λ will stand for the uniform distribution on S; $\lambda(dt) = (1/|S|)\sum_{s \in S} \delta_s(dt)$. Configuration spaces are defined as follows:

$$\Omega_\Lambda = \{\sigma = (\sigma_x)_{x \in \Lambda};\, \sigma_x \in S\}, \qquad \Lambda \subset \mathbf{Z}^d$$

$$\Omega = \Omega_{\mathbf{Z}^d}$$

For $\Lambda \subset \mathbf{Z}^d$ and $(\sigma, \omega) \in \Omega^2$, $\sigma_\Lambda \cdot \omega_{\Lambda^c}$ denotes the following configuration:

$$(\sigma_\Lambda \cdot \omega_{\Lambda^c})_x = \begin{cases} \sigma_x & \text{if} \quad x \in \Lambda \\ \omega_x & \text{if} \quad x \notin \Lambda \end{cases}$$

For $f\colon \Omega_\Lambda \to \mathbf{R}$ we introduce the notations

$$\nabla_X f(\sigma) = \int_{S^X} f(\xi_X \cdot \sigma_{X^c})\, \lambda^X(d\xi_X) - f(\sigma), \qquad X \subset\subset \mathbf{Z}^d$$

$$\|f\| = \sup_{\sigma \in \Omega_\Lambda} |f(\sigma)|$$

$$|||f||| = \sum_{x \in \Lambda} \|\nabla_x f\|$$

$$\operatorname{osc}_x(f) = \sup_{(\sigma, \sigma') \in \Omega^2} \{|f(\sigma) - f(\sigma')|;\, \sigma \equiv \sigma' \text{ off } x\}$$

$$\Delta_f = \{x \in \Lambda;\, f \text{ is not a constant with respect to } \sigma_x\}$$

The function spaces $\mathscr{C}$ and $\mathscr{C}_\Lambda$ $(\Lambda \subset \mathbf{Z}^d)$ are defined, respectively, by

$$\mathscr{C} = \{f\colon \Omega \to \mathbf{R};\, |\Delta_f| < \infty\}$$

$$\mathscr{C}_\Lambda = \{f\colon \Omega \to \mathbf{R};\, \Delta_f \subset \Lambda\}$$

The interaction and finite-volume Gibbs states. A family $\mathbf{\Phi} = \{\Phi_X \in \mathscr{C}_X\colon X \subset\subset \mathbf{Z}^d\}$ is called a *bounded, finite-range interaction* if it satisfies the following:

($\mathbf{\Phi}$-1) There exists $M_0 < \infty$ such that

$$\|\Phi\| := \sup_{x \in \mathbf{Z}^d} \sum_{X:\, X \ni x} \|\Phi_X\| \leqslant M_0 \tag{2.4}$$

(Φ-2) There exists $r_0 < \infty$ such that

$$r(\mathbf{\Phi}) := \sup\{\operatorname{diam}(X); \Phi_X \not\equiv 0\} \leqslant r_0 \tag{2.5}$$

$\|\Phi\|$ in (2.4) and $r(\mathbf{\Phi})$ in (2.5) are called the norm and the range of the interaction, respectively.

From here on we fix a bounded, finite-range interaction $\mathbf{\Phi}$. For each $\Lambda \subset\subset \mathbf{Z}^d$ we define the *Hamiltonian* $H_\Lambda \in \mathscr{C}$ by

$$H_\Lambda = \sum_{X:\, X \cap \Lambda \neq \varnothing} \Phi_X \tag{2.6}$$

For each $\Lambda \subset\subset \mathbf{Z}^d$ and $\omega \in \Omega$ we define the *finite-volume Gibbs state* $\mu^{\Lambda, \omega}$ as the probability measure on Ω_Λ in which each configuration $\sigma_\Lambda \in \Omega_\Lambda$ appears with probability

$$\mu^{\Lambda, \omega}(\{\sigma_\Lambda\}) = \frac{\exp - H_\Lambda(\sigma_\Lambda \cdot \omega_{\Lambda^c})}{Z^{\Lambda, \omega}} \tag{2.7}$$

where $Z^{\Lambda, \omega}$ is the normalizing constant.

The stochastic dynamics. We introduce now for the model above the time evolution called Glauber dynamics. We define for each $x \in \mathbf{Z}^d$ an operator $A_x: \mathscr{C} \to \mathscr{C}$ by

$$A_x f(\sigma) = \sum_{s \in S} c_x(\sigma, s)(f(\sigma \cdot^x s) - f(\sigma))$$

where $\sigma \cdot^x s$ is a configuration obtained from σ by replacing σ_x by s and the coefficient $c_x(\sigma, s)$, which is called the *flip rate*, is required to satisfy the following.

(R-1) *Boundedness*: There exist positive constants $\underline{c}$ and $\bar{c}$ such that

$$\underline{c} \leqslant c_x(\sigma, s) \leqslant \bar{c} \qquad \text{for all} \quad x \in \mathbf{Z}^d, \quad \sigma \in \Omega, \quad s \in S$$

(R-2) *Finite range*: There exists $r_1 \geqslant 0$ such that

$$\gamma(x, y) = 0 \qquad \text{if} \quad |x - y| > r_1 \tag{2.8}$$

where

$$\gamma(x, y) = \sup_{(\sigma, \sigma') \in \Omega^2} \left\{ \sum_{s \in S} |c_x(\sigma, s) - c_x(\sigma', s)|; \sigma \equiv \sigma' \text{ off } y \right\} \tag{2.9}$$

(R-3) *Detailed balance condition*: For all $x \in \mathbf{Z}^d$, $\sigma \in \Omega$, and $(s, s') \in S^2$,

$$\mu^{\{x\}, \sigma}(\{s\})\, c_x(\sigma \cdot^x s, s') = \mu^{\{x\}, \sigma}(\{s'\})\, c_x(\sigma \cdot^x s', s) \tag{2.10}$$

Remark 2.1. This is a very technical remark and is relevant only when one would like to identify a numerical constant M_1 which will appear in Theorem 3.2, Lemma 4.1, and Lemma 4.2. We set

$$M = \sup_x \sum_{y:\, y \neq x} \gamma(x, y)$$

$$\begin{aligned} M_1 &= \sum_{y \in \mathbf{Z}^d} \gamma_1(y) \\ &= (2r_1 + 1)^d\, |S|\, (\bar{c} - \underline{c}) \end{aligned} \tag{2.11}$$

where $\gamma(\cdot, \cdot)$ is defined by (2.9) and

$$\gamma_1(y) = \begin{cases} |S|\, (\bar{c} - \underline{c}) & \text{if } \ |y| \leqslant r_1 \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

The function $\gamma(\cdot, \cdot)$, together with the constant M plays fundamental role in analyzing a particle system (ref. 10, Chapter I). The function $\gamma_1(\cdot)$ is introduced as a shift-invariant object which bounds $\gamma(\cdot, \cdot)$ from above. Note that $\gamma(x, y) \leqslant \gamma_1(x - y)$ and hence $M \leqslant M_1$.

For fixed $\Lambda \subset\subset \mathbf{Z}^d$ and $\omega \in \Omega$ we define an operator $A^{\Lambda, \omega}: \mathscr{C}_\Lambda \to \mathscr{C}_\Lambda$ and the associated semigroup $(T_t^{\Lambda, \omega})_{t \geqslant 0}$ by

$$A^{\Lambda, \omega} f(\sigma) = \sum_{x \in \Lambda} A_x f(\sigma_\Lambda \cdot \omega_{\Lambda^c}), \qquad f \in \mathscr{C}_\Lambda$$

$$T_t^{\Lambda, \omega} = \exp tA^{\Lambda, \omega}$$

We then see from (2.10) that for all $\{f, g\} \subset \mathscr{C}_\Lambda$

$$\begin{aligned} \mathscr{E}^{\Lambda, \omega}(f, g) &\stackrel{\text{def}}{=} -\mu^{\Lambda, \omega}(f A^{\Lambda, \omega} g) \\ &= \tfrac{1}{2} \sum_{x \in \Lambda} \sum_{s \in S} \int \mu^{\Lambda, \omega}(d\sigma)\, c_x(\sigma, s)[f(\sigma \cdot^x s) - f(\sigma)] \\ &\quad \times [g(\sigma \cdot^x s) - g(\sigma)] \end{aligned} \tag{2.13}$$

We have defined $A^{\Lambda, \omega}$ and $T_t^{\Lambda, \omega}$ as operators acting on $\mathscr{C}_\Lambda$. But we will extend their domain of definition to $\mathscr{C}$ by applying them to "$\sigma \mapsto f(\sigma_\Lambda \cdot \omega_\Lambda)$."

The spectral gap, the logarithmic Sobolev constant, and the Dobrushin–Shlosman mixing condition. We now introduce a couple of quantities and a notion which plays crucial roles in analyzing the long-time behavior of the Glauber dynamics. We define the *inverse spectral gap* $\gamma_{SG}(\Lambda, \omega)$ as the smallest γ for which the following inequality is true for all $f \in \mathscr{C}$:

$$\mu^{\Lambda, \omega}(|f - \mu^{\Lambda, \omega} f|^2) \leqslant \gamma \mathscr{E}^{\Lambda, \omega}(f, f) \tag{2.14}$$

Similarly, we define the *logarithmic Sobolev constant* $\gamma_{LS}(\Lambda, \omega)$ as the smallest γ for which the following inequality is true for all $f \in \mathscr{C}$:

$$\mu^{\Lambda, \omega}\left(f^2 \log \frac{f^2}{\mu^{\Lambda, \omega}(f^2)}\right) \leqslant 2\gamma \mathscr{E}^{\Lambda, \omega}(f, f) \tag{2.15}$$

For $\mathscr{F} \subset \mathscr{A}$ an interaction Φ is said to satisfy the *Dobrushin–Shlosman mixing condition* over $\mathscr{F}$ if there exist constant $C_i \in (0, \infty)$ $(i = 1, 2)$ such that for all $\Lambda \in \mathscr{F}$, $y \notin \Lambda$, and $f \in \mathscr{C}_\Lambda$,

$$\operatorname{osc}_y(\mu^{\Lambda} f) \leqslant C_1 \, \|f\| \exp - \frac{d(y, \Delta_f)}{C_2} \tag{2.16}$$

In the sequel we will refer to the above condition as DSM($\mathscr{F}$).

It is convenient to introduce the following definition. We call a family $\{c(f) > 0\}_{f \in \mathscr{C}}$ an *admissible coefficient* if it satisfies

$$\sup_f \{c(f); \|f\| + \operatorname{diam} \Delta_f \leqslant m\} < \infty \qquad \text{for all} \quad m > 0 \tag{2.17}$$

What (2.17) requires is that $c(f)$ should have an upper bound in terms of $\|f\|$ and $\operatorname{diam} \Delta_f$, which are independent of where Δ_f is. Typically, const.$\|f\|$ or const.$|||f|||$ appears as the admissible coefficient $c(f)$.

Remark 2.2. We could have worked in the continuous spin setting, in which the discrete spin space S is replaced by a smooth, compact Riemannian manifold (in fact, C^2 is enough). What we have defined above can easily be modified so that they make sense in the continuous spin setting and the all results as well as their proofs in this paper remain valid up to the value of constants and boring technicalities.

3. THE RESULT

First we recall the following result:

Theorem 3.1.[16] Let $\mathscr{F}$ be either $\mathscr{A}$ or $\mathscr{B}_v(m)$ for arbitrarily fixed $v \in \mathbf{Z}^d$ and $m \geqslant 1$ [cf. (2.1), (2.3)]. Then each of the following conditions is equivalent to the DSM($\mathscr{F}$) [cf. (2.16)]:

(1a) The logarithmic Sobolev constant [cf. (2.15)] satisfies

$$\sup_{\Lambda \in \mathscr{F},\, \omega \in \Omega} \gamma_{\mathrm{LS}}(\Lambda, \omega) < \infty \tag{3.1}$$

(1b) The inverse spectral gap [cf. (2.14)] satisfies

$$\sup_{\Lambda \in \mathscr{F},\, \omega \in \Omega} \gamma_{\mathrm{SG}}(\Lambda, \omega) < \infty \tag{3.2}$$

(1c) There exist a constant $C > 0$ and an admissible coefficient $\{c(f) > 0\}_{f \in \mathscr{C}}$ [cf. (2.17)] such that for all $f \in \mathscr{C}$ and $t > 0$

$$\sup_{\Lambda \in \mathscr{F},\, \omega \in \Omega} \|T_t^{\Lambda, \omega} f - \mu^{\Lambda, \omega} f\| \leqslant c(f) \exp -\frac{t}{C} \tag{3.3}$$

Remark 3.1. In the original statement of this theorem [ref. 16, Theorem 1.8, part (c)] only the family $\mathscr{A}$ is under consideration. The statements for $\mathscr{B}_v(m)$ is implicit in the argument in that paper. The point is that both $\mathscr{A}$ and $\mathscr{B}_v(m)$ are closed under intersection and contain arbitrarily large cubes. The DSM condition restricted to boxes is easier to check than DSM($\mathscr{A}$) is. In fact, DSM($\mathscr{A}$) is known to be true for the Ising ferromagnet with $\beta < \beta_c/2$ or $|h| > 2d$ [ref. 5; see also ref. 3, (2.32) for the latter case], while the validity of DSM($\mathscr{B}_v(2)$) extends to $|h| > d-1$. As mentioned in ref. 6, Remark 2.3, there is also a difference between DSM($\mathscr{B}_v(1)$) and DSM($\mathscr{B}_v(m)$) with a large m if $d \geqslant 3$. The DSM condition based on cubes, together with an analogous statement to Theorem 3.1 in that context, is studied extensively in refs. 11–13 and 14. In this connection, it should be mentioned that ref. 11, Theorem 4.1, contains the following nice observation: if DSM($\bigcup_{v \in \mathbf{Z}^d} \bigcup_{m \geqslant m_0} \mathscr{C}_v(m)$) holds for some $m_0 \geqslant 1$, then DSM($\bigcup_{v \in \mathbf{Z}^d} \bigcup_{m \geqslant m_1} \mathscr{B}_v(m)$) holds for large enough $m_1 \geqslant 1$.

Next, we state the main result of this paper.

Theorem 3.2. Let $\mathscr{F}$ be either $\mathscr{A}$ or $\mathscr{B}_v(m)$ for arbitrarily fixed $v \in \mathbf{Z}^d$ and $m \geqslant 1$ [cf. (2.1), (2.3)]. Then each of the following conditions is equivalent to DSM($\mathscr{F}$) [cf. (2.16)]:

(2a) The logarithmic Sobolev constant [cf. (2.15)] satisfies

$$\varlimsup_{D \to \infty} \sup_{\Lambda \in \mathscr{F},\, \omega \in \Omega} \left\{ \frac{\gamma_{\mathrm{LS}}(\Lambda, \omega)}{\psi(\operatorname{diam} \Lambda)};\ \operatorname{diam} \Lambda \geqslant D \right\} < \frac{1}{40(d-1)\, r_1 M_1} \tag{3.4}$$

where $\psi(t) = t/\log t$. The constants r_1 and M_1 are given, respectively, by (2.8) and (2.11).

(2b) The inverse spectral gap [cf. (2.14)] satisfies

$$\varlimsup_{D\to\infty} \sup_{\Lambda\in\mathscr{F},\,\omega\in\Omega} \left\{\frac{\gamma_{\mathrm{SG}}(\Lambda,\omega)}{\psi(\operatorname{diam}\Lambda)};\ \operatorname{diam}\Lambda\geqslant D\right\} < \frac{1}{40(d-1)\,r_1 M_1} \tag{3.5}$$

where $\psi(t)=t/\log t$. The constants r_1 and M_1 are given, respectively, by (2.8) and (2.11).

(2c) There exists an admissible coefficient $\{c(f)>0\}_{f\in\mathscr{C}}$ [cf. (2.17)] and a nonincreasing function $\phi\colon(0,\infty)\to(0,\infty)$ such that $\phi(t)=o(t^{-2(d-1)})$ as $t\to\infty$ and such that

$$\sup_{\Lambda\in\mathscr{F},\,\omega\in\Omega} \mu^{\Lambda,\omega}\,|T_t^{\Lambda,\omega}f-\mu^{\Lambda,\omega}f|\leqslant c(f)\,\phi(t) \qquad \text{for all } f\in\mathscr{C} \tag{3.6}$$

(2d) There exist $C_1\in(2,\infty)$, an admissible coefficient $\{c(f)>0\}_{f\in\mathscr{C}}$ and a nonincreasing function $\phi\colon(0,\infty)\to(0,\infty)$ such that $\phi(t)=o(t^{-2(d-1)})$ as $t\to\infty$ and such that

$$\operatorname{osc}_y(\mu^{\Lambda,\cdot}f)\leqslant c(f)\,\phi(d(y,\Delta_f)) \tag{3.7}$$

for all $\Lambda\in\mathscr{F}$, $f\in\mathscr{C}_\Lambda$, and $y\in\partial_{r_0}\Lambda$ which satisfy $\operatorname{diam}\Lambda\leqslant C_1 d(y,\Delta_f)$.

(2e) There exists a nonincreasing function $\phi\colon(0,\infty)\to(0,\infty)$ such that $\phi(t)=o(t^{-(d-1)})$ as $t\to\infty$ and such that

$$\operatorname{osc}_y(\mu^{\Lambda,\cdot}f)\leqslant \|f\|\,\phi(d(y,\Delta_f)) \tag{3.8}$$

for all $\Lambda\in\mathscr{F}$, $f\in\mathscr{C}_\Lambda$, and $y\in\partial_{r_0}\Lambda$.

Remark 3.2. It follows from Theorem 3.2 that the condition (2a) or even (2b) implies (1c) in Theorem 3.1. This is reminiscent of ref. 8, Theorem 5.3. Although (2a) and (2b) require control over boxes, not only over cubes, the conclusion (1c) is quite rewarding as compared with that of the above-quoted reference. The bound $1/\{40(d-1)\,r_1 M_1\}$ on the right-hand sides of (3.4), (3.5) is technical and is in no way sharp.

Remark 3.3. It follows from Theorem 3.2 that (2c) implies (1c), which roughly says the following: if the rate of convergence of a Glauber dynamics is bounded by a certain negative power of the time t, then the convergence is necessarily exponentially fast in t. As is mentioned in the Introduction, this is not the first time a result of this kind has been obtained. For example, it is shown in ref. 15, Theorem 3.6, that L^∞-convergence as fast as $t^{-(2d+\varepsilon)}$ implies exponentially fast L^∞-convergence. Also, it is mentioned in ref. 9, Remark 4.3, that L^2-convergence as fast as $t^{-(2d+\varepsilon)}$ implies exponentially fast L^2-convergence. On the other hand, our result [(2c)⇒(1c)] says that L^1-convergence faster than $t^{-2(d-1)}$ in the sense of (2c) is enough for exponentially fast L^∞-convergence, which improves two results mentioned above. It should also be mentioned that in the case $S=\{-1, +1\}$ and the flip rate is attractive, (1c) is also implied by the following condition:

$$\lim_{t \to \infty} t^d \sup_{\Lambda \in \mathscr{F}, \omega \in \Omega} \sup_{x \in \Lambda} \{(T_t^{\Lambda,\omega}\sigma_x)(+)-(T_t^{\Lambda,\omega}\sigma_x)(-)\}=0 \tag{3.9}$$

where $(\pm)$ stand for configurations with all spins equal to ± 1. This follows from the proof of ref. 1, Theorem 4, with a slight adjustment in order to circumvent the lack of shift invariance. In $d=2$ condition (2c) is milder than (3.9) (see ref. 1, Lemma 2.1).

Remark 3.4. Condition (2e) says that the exponential decay in the definition of DSM($\mathscr{F}$) is equivalent to a certain polynomial decay. Technically, condition (2d) plays the role of a junction between a statement in terms of dynamics, like (2b) or (2c), and that in terms of equilibrium, like (2e) (cf. Lemmas 4.2 and 4.3). Let us compare condition (2d) with a similar condition in ref. 4. In ref. 4 it is proved that DSM($\mathscr{F}$) has the following relaxed criterion, which is called condition (IIIb) in that paper: there exists a nonincreasing function ϕ: $(0, \infty) \to (0, \infty)$ such that $\phi(t)=o(t^{-2(d-1)})$ as $t \to \infty$ and such that

$$\operatorname{osc}_y(\mu^{\Lambda,\cdot} f) \leqslant \|f\| \sum_{x \in \Delta_f} \phi(|x-y|) \tag{3.10}$$

for all $\Lambda \in \mathscr{F}$, $f \in \mathscr{C}_\Lambda$, and $y \in \partial_{r_0}\Lambda$. As can easily be seen, condition (2d) relaxes condition (IIIb) even further.

4. PROOF OF THEOREM 3.2

Lemma 4.1. Let $\Lambda \subset\subset \mathbf{Z}^d$ and $\omega \in \Omega$ be fixed. Suppose that $\phi(t)$ is a nonincreasing function such that $\lim_{t \to \infty} \phi(t)=0$ and that

$$\mu^{\Lambda,\omega}\, |T_t^{\Lambda,\omega} f-\mu^{\Lambda,\omega} f| \leqslant c(f)\, \phi(t) \qquad \text{for all } f \in \mathscr{C} \text{ and } t>0 \tag{4.1}$$

with some coefficient $c(f)$. Then there exists $C \in (0, \infty)$ which depends only on the flip rate such that for all $\{f, g\} \subset \mathscr{C}$

$$\begin{aligned} \mu^{\Lambda,\omega}(f; g) &\stackrel{\text{def}}{=} \mu^{\Lambda,\omega}(fg)-\mu^{\Lambda,\omega} f \mu^{\Lambda,\omega} g \\ &\leqslant c(f, g)\, \phi\left(\frac{d(\Delta_f, \Delta_g)}{10 M_1 r_1}\right)+C\, |||f||| \cdot |||g|||\, \exp -\frac{d(\Delta_f, \Delta_g)}{10 r_1} \end{aligned} \tag{4.2}$$

where $c(f, g) = c(fg) + \|f\| \, c(g) + c(f) \, \|g\|$. The constants r_1 and M_1 are given respectively by (2.8) and (2.11).

Proof. We first bound the left-hand side of (4.2) by three terms:

$$\mu^{\Lambda, \omega} \, |T_t^{\Lambda, \omega}(fg) - T_t^{\Lambda, \omega} f \cdot T_t^{\Lambda, \omega} g|$$

$$\leqslant \mu^{\Lambda, \omega} \, |T_t^{\Lambda, \omega}(fg) - T_s^{\Lambda, \omega}(fg)| \tag{4.3}$$

$$+ \mu^{\Lambda, \omega} \, |T_s^{\Lambda, \omega}(fg) - T_s^{\Lambda, \omega} f \cdot T_s^{\Lambda, \omega} g| \tag{4.4}$$

$$+ \mu^{\Lambda, \omega} \, |T_s^{\Lambda, \omega} f \cdot T_s^{\Lambda, \omega} g - T_t^{\Lambda, \omega} f \cdot T_t^{\Lambda, \omega} g| \tag{4.5}$$

The condition (4.1) is available to bound the first and the third terms:

$$(4.3) \leqslant \mu^{\Lambda, \omega} \, |T_t^{\Lambda, \omega}(fg) - \mu^{\Lambda, \omega}(fg)| + \mu^{\Lambda, \omega} \, |T_s^{\Lambda, \omega}(fg) - \mu^{\Lambda, \omega}(fg)|$$

$$\leqslant c(fg)(\phi(t) + \phi(s)) \tag{4.6}$$

Similarly,

$$(4.5) \leqslant (\|f\| \, c(g) + c(f) \, \|g\|)(\phi(t) + \phi(s)) \tag{4.7}$$

We now apply Proposition 4.18 of ref. 10, p. 40, to (4.4) to see that

$$(4.4) \leqslant C \, |||f||| \cdot |||g||| \exp(4M_1 s - \delta\rho) \tag{4.8}$$

where $\delta = 1/2r_1$, $\rho = d(\Delta_f, \Delta_g)$, and $C > 0$ is a constant which depends only on the flip rate (cf. Remark 4.1 below). At this point we take $s = \delta\rho/5M_1$. We then have by (4.6)–(4.8) that

$$\mu^{\Lambda, \omega} \, |T_t^{\Lambda, \omega}(fg) - T_t^{\Lambda, \omega} f \cdot T_t^{\Lambda, \omega} g|$$

$$\leqslant (c(fg) + \|f\| \, c(g) + c(f) \, \|g\|)(\phi(t) + \phi(\delta\rho/5M_1))$$

$$+ C \, |||f||| \cdot |||g||| \exp -\frac{\delta\rho}{5}$$

which proves (4.2) by letting $t \to \infty$. QED

Remark 4.1. Since $(T_t^{\Lambda, \omega})_{t \geqslant 0}$ is not a shift-invariant particle system, (4.8) does not come from a direct application of ref. 10, Proposition 4.18, but from a modification of the proof of that proposition, to be precise. The modification is simply to replace $\gamma(x, y)$ by its upper bound $\gamma_1(x - y)$ (cf. Remark 2.1). The choice $\delta = 1/2r_1$ is possible for the following reason. Going back to the proof of the proposition, δ has been chosen so that

$$\sum_{x \in \mathbf{Z}^d} \gamma_1(x) \exp(\delta\, |x|) \leqslant 2M_1 \tag{4.9}$$

Since the left-hand side of (4.9) is not greater than $\exp(\delta r_1)\, M_1$, our choice $\delta = 1/2r_1$ is more than enough.

Lemma 4.2. Let $C_0 \in [0, \infty)$, $C_1 \in (0, \infty)$, and $\mathscr{F} \subset \mathscr{A}$ be arbitrary. Suppose that there exists $C_2 \in (0, \infty)$ such that

$$\varlimsup_{D \to \infty} \sup_{\Lambda \in \mathscr{F},\, \omega \in \Omega} \left\{ \frac{\gamma_{\mathrm{SG}}(\Lambda, \omega)}{\psi(\operatorname{diam} \Lambda)};\ \operatorname{diam} \Lambda \geqslant D \right\} < C_2 \tag{4.10}$$

where $\psi(t) = t/\log t$. Then there exists $C_3 \in (0, \infty)$ such that

$$\operatorname{osc}_y(\mu^{\Lambda, \cdot} f) \leqslant C_3\, |||f|||\, d(y, \Delta_f)^{-1/(5C_1C_2M_1r_1)} \tag{4.11}$$

for all $\Lambda \in \mathscr{F}$, $f \in \mathscr{C}_\Lambda$, and $y \in \partial_{r_0} \Lambda$ which satisfy

$$\operatorname{diam} \Lambda \leqslant C_1\, d(y, \Delta_f) + C_0 \tag{4.12}$$

The constants r_1 and M_1 are given by (2.8) and (2.11), respectively.

Proof. By (4.10), there exist $\theta \in (0, 1)$ and $D \geqslant 1$ such that

$$\frac{\sup_{\omega \in \Omega} \gamma_{\mathrm{SG}}(\Lambda, \omega)}{C_2 \psi(\operatorname{diam} \Lambda)} \leqslant \theta \leqslant \frac{C_1 D}{C_1 D + C_0} \tag{4.13}$$

for any $\Lambda \in \mathscr{F}$ with $\operatorname{diam} \Lambda \geqslant D$. To prove (4.11), we may assume $\rho := d(y, \Delta_f) \geqslant D + r_0$, since the left-hand side of (4.11) is not greater than $|||f|||$ in any case. Let $(\omega, \omega') \in \Omega^2$ be such that $\omega' \equiv \omega$ off y. We set $g = d\mu^{\Lambda, \omega'}/d\mu^{\Lambda, \omega}$. Note that

$$\|g\| \leqslant \exp(4\, \|\Phi\|) \qquad \text{and} \qquad \Delta_g \subset \Lambda \cap \partial_{r_0}\{y\}$$

Using these and Lemma 4.1, we have

$$|(\mu^{\Lambda, \omega'} - \mu^{\Lambda, \omega}) f| = |\mu^{\Lambda, \omega}(g; f)|$$

$$\leqslant c_1\, |||f|||\, \exp - \frac{\rho - r_0}{10 M_1 r_1 \gamma_{\mathrm{SG}}(\Lambda, \omega)} \tag{4.14}$$

$$+ c_2\, |||f|||\, \exp - \frac{\rho - r_0}{10 r_1} \tag{4.15}$$

Here and in what follows c_i ($i=1, 2,\dots$) stand for constants which are independent of Λ, f, and y. Since $\operatorname{diam} \Lambda \geqslant \rho - r_0 \geqslant D$, we can take advantage of (4.13) and (4.12) to bound (4.14) from above as follows:

$$
\begin{aligned}
(4.14) &\leqslant c_3 \,|||f|||\, \exp\left(-\frac{\rho}{10M_1 r_1 \theta C_2(C_1\rho + C_0)} \log(C_1\rho + C_0)\right) \\
&\leqslant c_3 \,|||f|||\, \exp\left(-\frac{1}{10C_1 C_2 M_1 r_1} \log(C_1\rho + C_0)\right) \\
&\leqslant c_3 \,|||f|||\, \rho^{-1/(5C_1C_2M_1r_1)} \qquad (4.16)
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
(4.15) &\leqslant c_4 \,|||f|||\, \exp -\frac{\rho}{10r_1} \\
&\leqslant c_5 \,|||f|||\, \rho^{-1/(10C_1C_2M_1r_1)} \qquad (4.17)
\end{aligned}
$$

Putting (4.16) and (4.17) together, we conclude (4.11). QED

Lemma 4.3. Let $C_1 \in (2, \infty)$ and $n > 1$ be such that $n(4n-3)/2(n-1)^2 \leqslant C_1$ and $\mathscr{F} = \mathscr{B}_v(m)$ for arbitrarily fixed $v \in \mathbf{Z}^d$ and $m \geqslant 1$. Suppose that there exist an admissible coefficient $\{c(f) > 0\}_{f \in \mathscr{C}}$ and a nonincreasing function ϕ: $(0, \infty) \to (0, \infty)$ such that

$$\operatorname{osc}_y(\mu^{\Lambda,\cdot} f) \leqslant c(f)\, \phi(d(y, \Delta_f)) \qquad (4.18)$$

for all $\Lambda \in \mathscr{F}$, $f \in \mathscr{C}_\Lambda$, and $y \in \partial_{r_0}\Lambda$ which satisfy $\operatorname{diam} \Lambda \leqslant C_1 d(y, \Delta_f)$. Then there exists $C_2 \in (0, \infty)$ such that

$$\operatorname{osc}_y(\mu^{\Lambda,\cdot} f) \leqslant C_2 \,\|f\|\, d(y, \Delta_f)^{d-1}\, \phi(d(y, \Delta_f) - 2m - r_0) \qquad (4.19)$$

for all $\Lambda \in \mathscr{F}$, $f \in \mathscr{C}_\Lambda$, and $y \in \partial_{r_0}\Lambda$ which satisfy $d(y, \Delta_f) \geqslant n(2m + r_0)$.

For $\mathscr{F} = \mathscr{A}$ the same statement with $m = 1$ is true.

Proof. Take $\Lambda \in \mathscr{F}$, $f \in \mathscr{C}_\Lambda$, and $y \in \partial_{r_0}\Lambda$, which satisfy $\rho := d(y, \Delta_f) \geqslant n(2m + r_0)$. We are going to apply our assumption to $\Lambda \cap \Gamma$ rather than Λ itself, where Γ is an element of $\mathscr{C}_v(m)$ we now define. We define $\Gamma_k \in \mathscr{C}_v(m)$ ($k = 1, 2,\dots$) by

$$\Gamma_k = \{x \colon d(x, \Gamma_0) \leqslant km\}$$

where Γ_0 is an element in $\mathscr{C}_v(m)$ such that $\operatorname{diam} \Gamma_0 = m$ and $y \in \Gamma_0$. We set $\Gamma = \Gamma_K$, where $K = \max\{k \geqslant 1 \colon \Gamma_k \cap \Delta_f = \varnothing\}$. We then have by definition that

$$Km \leqslant \rho \leqslant (K+2)\, m \qquad (4.20)$$

and hence that

$$K \geqslant \frac{\rho}{m} - 2 \geqslant 2(n-1) \tag{4.21}$$

For fixed $(\omega, \omega') \in \Omega^2$ with $\omega \equiv \omega'$ off y we set $g = d\mu^{\Lambda, \omega'}/d\mu^{\Lambda, \omega}$. Note that

$$\|g\| \leqslant \exp(4\,\|\Phi\|), \qquad \Delta_g \subset \Lambda \cap \partial_{r_0}\{y\} \tag{4.22}$$

and that for $z \in \Lambda \cap \partial_{r_0}\Gamma$

$$\operatorname{diam}(\Gamma \cap \Lambda) \leqslant C_1\, d(z, \Delta_g) \tag{4.23}$$

The first two observations are standard. The third one can be seen as follows. We have by (4.22) and (4.20) that

$$\begin{aligned} d(z, \Delta_g) &\geqslant |z - y| - r_0 \\ &\geqslant Km - r_0 \\ &\geqslant \left(\frac{\rho}{m} - 2\right) m - r_0 \\ &= \rho - 2m - r_0 \\ &\geqslant \frac{n-1}{n}\rho \end{aligned} \tag{4.24}$$

By (4.20), (4.21), and (4.24), we get

$$\begin{aligned} \operatorname{diam}(\Gamma \cap \Lambda) &\leqslant (2K+1)\, m \\ &\leqslant \left(2 + \frac{1}{K}\right)\rho \\ &\leqslant \left(2 + \frac{1}{2(n-1)}\right)\frac{n}{n-1}\, d(z, \Delta_g) \\ &\leqslant C_1\, d(z, \Delta_g) \end{aligned}$$

The proof of (4.19) comes down to the following estimate:

$$\|\mu^{\Gamma \cap \Lambda, \cdot}(g-1)\| \leqslant C_2 \rho^{d-1} \phi(\rho - 2m - r_0) \tag{4.25}$$

In fact,

$$|(\mu^{\Lambda,\omega'}-\mu^{\Lambda,\omega})f| = |\mu^{\Lambda,\omega}[f(g-1)]|$$
$$\leqslant \|f\|\cdot\|\mu^{\Gamma\cap\Lambda,\cdot}(g-1)\|$$

On the other hand, (4.25) can be seen as follows:

$$\begin{aligned}|\mu^{\Gamma\cap\Lambda,\xi}(g-1)| &= |\mu^{\Lambda,\omega}(\mu^{\Gamma\cap\Lambda,\xi}g-\mu^{\Gamma\cap\Lambda,\cdot}g)|\\ &\leqslant \sup_{(\xi,\eta)\in\Omega^2}\{|\mu^{\Gamma\cap\Lambda,\xi}g-\mu^{\Gamma\cap\Lambda,\eta}g|;\ \xi\equiv\eta \text{ outside } \Lambda\}\\ &\leqslant |\Lambda\cap\partial_{r_0}\Gamma| \sup_{z\in\Lambda\cap\partial_{r_0}\Gamma} \operatorname{osc}_z(\mu^{\Gamma\cap\Lambda,\cdot}g)\\ &\leqslant c_1\rho^{d-1}c(g)\sup_{z\in\Lambda\cap\partial_{r_0}\Gamma}\phi(d(z,\Delta_g))\\ &\leqslant c_2\rho^{d-1}\phi(\rho-2m-r_0)\end{aligned}$$

The first equality comes from an identity: $1=\mu^{\Lambda,\omega}g=\mu^{\Lambda,\omega}(\mu^{\Gamma\cap\Lambda,\cdot}g)$. The inequality in the fourth line is an application of (4.18) to $\Gamma\cap\Lambda\in\mathscr{F}$, which is allowed by (4.23) under our present assumption. To proceed to the last line, we used (4.22). QED

Lemma 4.4. Let $\mathscr{F}$ be either $\mathscr{A}$ or $\mathscr{B}_v(m)$ for arbitrarily fixed $v\in\mathbf{Z}^d$ and $m\geqslant 1$. Suppose that (2e) in Theorem 3.2 holds and set $\alpha_{x,y}=\phi(|x-y|)$. Then,

$$\lim_{L\to\infty}\frac{1}{L^d}\sum_{x\in[0,L)^d}\ \sum_{y\in\partial_{r_0}[0,L)^d}\alpha_{x,y}=0 \tag{4.26}$$

and for all $X\in\mathscr{F}$ and $y\notin X$

$$\|\nabla_y\mu^{X,\cdot}f-\mu^{X,\cdot}\nabla_y f\|\leqslant\sum_{x\in X}\|\nabla_x f\|\,\alpha_{x,y} \tag{4.27}$$

Remark 4.2. A mixing property defined by the set of conditions (4.26) and (4.27) is essentially the same as the condition (GS2) in ref. 7 and the condition DSM(Y) in ref. 16. The condition turns out to be equivalent to DSM($\mathscr{F}$) in the end (cf. the proof of Theorem 3.2).

Proof. It is easy to see (4.26). In fact,

$$\sum_{x \in [0, L)^d} \sum_{y \in \partial_{r_0}[0, L)^d} \alpha_{x, y} = \sum_{x \in [0, L)^d} \sum_{y \in \partial_{r_0}[0, L)^d} \phi(|x-y|)$$
$$\leqslant \sum_{m=1}^{L+r_0} \sum_{y \in \partial_{r_0}[0, L)^d} \sum_{x: |x-y|=m} \phi(m)$$
$$\leqslant c_1 L^{d-1} \sum_{m=1}^{L+r_0} m^{d-1} \phi(m)$$

and thus

$$\frac{1}{L^d} \sum_{x \in [0, L)^d} \sum_{y \in \partial_{r_0}[0, L)^d} \alpha_{x, y} \leqslant \frac{c_1}{L} \sum_{m=1}^{L+r_0} m^{d-1} \phi(m)$$
$$\to 0 \qquad \text{as} \quad L \to \infty$$

We next prove (4.27). Take $X \in \mathscr{F}$ and an alignment $\{x_m\}_{m=1}^n$ of points in X. We set $I_m = \{x_j\}_{j=1}^{m-1}$ and $J_m = \{x_j\}_{j=m}^n$. Note first that

$$\nabla_X f(\sigma) = \sum_{m=1}^{n} \int \lambda^{I_m}(d\xi_{I_m}) \, \nabla_{x_m} f(\xi_{I_m} \cdot \sigma_{I_m^c})$$

Using this expression, we have that

$$\nabla_y \mu^{X, \omega} f - \mu^{X, \omega} \nabla_y f$$
$$= \int \lambda(ds)(\mu^{X, \omega \cdot s} - \mu^{X, \omega})(d\sigma_X) \, f(\sigma_X \cdot (\omega \cdot s)_{X^c})$$
$$= -\int \lambda(ds)(\mu^{X, \omega \cdot s} - \mu^{X, \omega})(d\sigma_X) \, \nabla_X f(\sigma_X \cdot (\omega \cdot s)_{X^c})$$
$$= -\sum_{m=1}^{n} \int \lambda(ds) \int (\mu^{X, \omega \cdot s} - \mu^{X, \omega})(d\sigma_X) \, \overline{f_m}(\sigma_X \cdot (\omega \cdot s)_{X^c}) \quad (4.28)$$

where $\omega \cdot s$ stands for a configuration obtained from σ by replacing σ_y by s and

$$\overline{f_m}(\sigma) = \int \lambda^{I_m}(d\xi_{I_m}) \, \nabla_{x_m} f(\xi_{I_m} \cdot \sigma_{I_m^c})$$

At this point, fix y and choose the alignment so that $|x_j - y|$ is nondecreasing in $j = 1, 2,\ldots$. Since $\|\overline{f_m}\| \leqslant \|\nabla_{x_m} f\|$ and "$\sigma_X \mapsto \overline{f_m}(\sigma_X \cdot (\omega \cdot s)_{X^c})$" $\in \mathscr{C}_{J_m}$, we have by (4.28) and (2e) that

$$\begin{aligned}\|\nabla_y \mu^{X.} f - \mu^{X.} \nabla_y f\| &\leqslant \sum_{m=1}^{n} \|\nabla_{x_m} f\| \,\phi(d(y, J_m)) \\ &= \sum_{m=1}^{n} \|\nabla_{x_m} f\| \,\phi(|x_m - y|) \\ &= \sum_{x \in X} \|\nabla_x f\| \,\alpha_{x,y}\end{aligned}$$

This proves (4.27). QED

Proof of Theorem 3.2. In view of Theorem 3.1, all conditions in Theorem 3.2 are obviously necessary for the Dobrushin–Shlosman mixing condition (2.16) to be true. To show that (2a), (2b), (2d), and (2e) are sufficient, we prove the following sequence of implications:

$$(2a) \Rightarrow (2b) \Rightarrow (2d) \Rightarrow (2e) \Rightarrow (1b) \tag{4.29}$$

The condition (1b) is equivalent to the Dobrushin–Shlosman mixing condition by Theorem 3.1. The condition (2a) immediately implies (2b), since $\gamma_{\mathrm{SG}} \leqslant \gamma_{\mathrm{LS}}$ [ref. 2, p. 224, (6.1.7)]. Implication from (2b) to (2d) is a consequence of Lemma 4.2, and (2d) implies (2e) by Lemma 4.3. Now suppose that (2e) holds. This means that we may assume the conclusion of Lemma 4.4. Then, by the argument used in Section 2 of ref. 16, we obtain (1b) in Theorem 3.1. See the derivation of Corollary 2.8 in that paper. This completes the proof of (4.29). On the other hand, Lemma 4.1 says that (2c) implies (2d), and hence also (1b) by (4.29). QED

ACKNOWLEDGMENTS

This work originated in conversations with Yasunari Higuchi, whom I would like to thank. I am also indebted to Roberto Schonmann for stimulating discussions through which the improvement of a previous version of this paper was incubated. A question which Tetsuya Hattori asked me about the difference among DSM($\mathscr{F}$) for different $\mathscr{F}$ gave me another incentive to reconsider the framework of the previous version. Finally, I am grateful for Dan Stroock for his interest in this work and for hospitality at MIT.

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